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Integrable Kondo impurities in the one-dimensional supersymmetric extended Hubbard model

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Abstract. An integrable Kondo problem in the one-dimensional supersymmetric extended Hubbard model is studied by means of the boundary graded quantum inverse scattering method. The boundary K-matrices depending on the local moments of the impurities are presented as a non-trivial realization of the graded reflection equation algebras in a two-dimensional impurity Hilbert space. Further, the model is solved by using the algebraic Bethe ansatz method and the Bethe ansatz equations are obtained.

The Kondo problem describing the effect due to the exchange interaction between the magnetic impurity and the conduction electrons plays a very important role in condensed matter physics [1]. Wilson [2] developed a very powerful numerical renormalization group approach, and the model was also solved by the coordinate Bethe ansatz method [3, 4] which gives the specific heat and magnetization. More recently, a conformal field theory approach was developed by Affleck and Ludwig [5] based on a previous work by Nozières [6]. In the conventional Kondo problem, the interaction between the conduction electrons is discarded, due to the fact that the interacting electron system can be described as a Fermi liquid. Recently, much attention has been paid to the study of the theory of magnetic impurities in Luttinger liquids (see, e.g., [7, 8]). Although some powerful methods, such as the bosonization method, boundary conformal field theory and the density matrix renormalization group method, are available to help us gain an understanding of the critical behaviour of Kondo impurities coupled to a Fermi or Luttinger liquid, some simple integrable models which allow exact solutions are still desirable.

Several integrable magnetic or non-magnetic impurity problems describing a few impurities embedded in some correlated electron systems have so far appeared in the literature. Among them are several versions of the supersymmetric t–J model with impurities [9–12]. Such an idea to incorporate an impurity into a closed chain may date back to Andrei and Johannesson [13] (see also [14, 15]). However, the model thus constructed suffers a lack of backward scattering and results in a very complicated Hamiltonian which is difficult to justify on physical grounds. Therefore, as observed by Kane and Fisher [16], it seems to be advantageous to adopt open boundary conditions with the impurities situated at the ends of the chain when studying Kondo impurities coupled to integrable strongly correlated electron systems [17–19].

In this paper, an integrable Kondo problem in the one-dimensional (1D) supersymmetric extended Hubbard model is studied. It should be emphasized that the new non-c-number boundary K-matrices arising from our approach are highly non-trivial, in the sense that they cannot be factorized into the product of a c-number boundary K-matrix and the corresponding local monodromy matrices. The model is solved by means of the algebraic Bethe ansatz method and the Bethe ansatz equations are derived.

Let $c_{j,\sigma}$ and $c_{j,\sigma}^\dagger$ denote electronic creation and annihilation operators for spin σ at site j, which satisfy the anti-commutation relations $\{c_{i,\sigma}^\dagger, c_{j,\tau}\} = \delta_{ij}\delta_{\sigma\tau}$, where $i, j = 1, 2, \ldots, L$ and $\sigma, \tau = \uparrow, \downarrow$. We consider the following Hamiltonian which describes two impurities coupled to the supersymmetric extended Hubbard open chain:

$$\begin{aligned} H = &-\sum_{j=1,\sigma}^{L-1}(c_{j,\sigma}^\dagger c_{j+1,\sigma} + \text{h.c.})(1 - n_{j,-\sigma} - n_{j+1,-\sigma}) \\ &- \sum_{j=1}^{L-1}(c_{j,\uparrow}^\dagger c_{j,\downarrow}^\dagger c_{j+1,\downarrow} c_{j+1,\uparrow} + \text{h.c.}) + 2\sum_{j=1}^{L-1}\left(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} - \tfrac{1}{4} n_j n_{j+1}\right) \\ &+ J_a \boldsymbol{S}_1 \cdot \boldsymbol{S}_a + V_a n_1 + U_a n_{1\uparrow} n_{1\downarrow} + J_b \boldsymbol{S}_L \cdot \boldsymbol{S}_b + V_b n_L + U_b n_{L\uparrow} n_{L\downarrow} \end{aligned} \tag{1}$$

where J_g, V_g and U_g $(g = a, b)$ are the Kondo coupling constants, the impurity scalar potentials and the boundary Hubbard-like interaction constants, respectively; $\boldsymbol{S}_j = \frac{1}{2}\sum_{\sigma,\sigma'} c_{j\sigma}^\dagger \boldsymbol{\sigma}_{\sigma\sigma'} c_{i\sigma'}$ is the spin operator of the conduction electrons; $\boldsymbol{S}_g (g = a, b)$ are the local moments with spin-$\frac{1}{2}$ located at the left and right ends of the system, respectively; $n_{j\sigma}$ is the number density operator $n_{j\sigma} = c_{j\sigma}^\dagger c_{j\sigma}$, $n_j = n_{j\uparrow} + n_{j\downarrow}$.

The supersymmetry algebra underlying the bulk Hamiltonian of this model is $gl(2|2)$, and the integrability of the model on a closed chain has been studied extensively by Essler *et al* [20]. It is quite interesting to note that although the introduction of the impurities spoils the supersymmetry, there is still a remaining $u(2) \otimes u(2)$ symmetry in the Hamiltonian (1). As a result, one may add some terms like the Hubbard interaction $U\sum_{j=1}^L n_{j\uparrow} n_{j\downarrow}$, the chemical potential term $\mu\sum_{j=1}^L n_j$ and the external magnetic field $h\sum_{j=1}^L (n_{j\uparrow} - n_{j\downarrow})$ to the Hamiltonian (1), without spoiling the integrability. This explains why the model is so named (also called the EKS model). Below we will establish the quantum integrability of the Hamiltonian (1) for a special choice of the model parameters J_g, V_g and U_g

$$J_g = -\frac{2}{c_g(c_g + 2)} \qquad V_g = -\frac{2c_g^2 + 2c_g - 1}{2c_g(c_g + 2)} \qquad U_g = -\frac{1 - c_g^2}{c_g(c_g + 2)}. \tag{2}$$

This is achieved by showing that it can be derived from the (graded) boundary quantum inverse scattering method [21, 22].

Let us recall that the Hamiltonian of the 1D supersymmetric extended Hubbard model with periodic boundary conditions commutes with the transfer matrix, which is the supertrace of the monodromy matrix $T(u) = R_{0L}(u) \cdots R_{01}(u)$. Here the quantum R-matrix $R_{0j}(u)$ takes the form

$$R = \frac{u - 2P}{u - 2} \tag{3}$$

where u is the spectral parameter, P denotes the graded permutation operator and the subscript 0 denotes the four-dimensional (4D) auxiliary superspace $V = C^{2,2}$ with the grading $[i] = 0$ if $i = 1, 2$ and 1 if $i = 3, 4$. It should be noted that the supertrace is carried out for the auxiliary superspace V. The elements of the supermatrix $T(u)$ are the generators of an associative superalgebra $\mathcal{A}$ defined by the relations

$$R_{12}(u_1 - u_2)\overset{1}{T}(u_1)\overset{2}{T}(u_2) = \overset{2}{T}(u_2)\overset{1}{T}(u_1)R_{12}(u_1 - u_2) \tag{4}$$

where $\overset{1}{X} \equiv X \otimes 1$, $\overset{2}{X} \equiv 1 \otimes X$ for any supermatrix $X \in End(V)$. For later use, we list some useful properties enjoyed by the R-matrix: (a) unitarity, $R_{12}(u)R_{21}(-u) = \rho(u)$ and (b) crossing-unitarity, $R_{12}^{st_2}(-u)R_{21}^{st_2}(u) = \tilde{\rho}(u)$ with $\rho(u)$, $\tilde{\rho}(u)$ being some scalar functions.

In order to describe integrable electronic models on open chains, we introduce two associative superalgebras $\mathcal{T}_-$ and $\mathcal{T}_+$ defined by the R-matrix $R(u_1 - u_2)$ and the relations [21, 22]

$$R_{12}(u_1 - u_2)\overset{1}{\mathcal{T}}_-(u_1)R_{21}(u_1 + u_2)\overset{2}{\mathcal{T}}_-(u_2) = \overset{2}{\mathcal{T}}_-(u_2)R_{12}(u_1 + u_2)\overset{1}{\mathcal{T}}_-(u_1)R_{21}(u_1 - u_2) \tag{5}$$

$$\begin{aligned} &R_{21}^{st_1 ist_2}(-u_1 + u_2)\overset{1}{\mathcal{T}}{}_+^{st_1}(u_1)R_{12}(-u_1 - u_2)\overset{2}{\mathcal{T}}{}_+^{ist_2}(u_2) \\ &\qquad = \overset{2}{\mathcal{T}}{}_+^{ist_2}(u_2)R_{21}(-u_1 - u_2)\overset{1}{\mathcal{T}}{}_+^{st_1}(u_1)R_{12}^{st_1 ist_2}(-u_1 + u_2) \end{aligned} \tag{6}$$

respectively. Here the supertransposition st_α $(\alpha = 1, 2)$ is only carried out in the αth factor superspace of $V \otimes V$, whereas ist_α denotes the inverse operation of st_α. By modifying Sklyanin's arguments [23], one may show that the quantities $\tau(u)$ given by $\tau(u) = str(\mathcal{T}_+(u)\mathcal{T}_-(u))$ constitute a commutative family, i.e. $[\tau(u_1), \tau(u_2)] = 0$.

One can obtain a class of realizations of the superalgebras $\mathcal{T}_+$ and $\mathcal{T}_-$ by choosing $\mathcal{T}_\pm(u)$ to be the form

$$\mathcal{T}_-(u) = T_-(u)\tilde{\mathcal{T}}_-(u)T_-^{-1}(-u) \qquad \mathcal{T}_+^{st}(u) = T_+^{st}(u)\tilde{\mathcal{T}}_+^{st}(u)\big(T_+^{-1}(-u)\big)^{st} \tag{7}$$

with

$$T_-(u) = R_{0M}(u)\cdots R_{01}(u) \qquad T_+(u) = R_{0L}(u)\cdots R_{0,M+1}(u) \qquad \tilde{\mathcal{T}}_\pm(u) = K_\pm(u) \tag{8}$$

where $K_\pm(u)$, called boundary K-matrices, are representations of $\mathcal{T}_\pm$ in some representation superspace. Although many attempts have been made to find c-number boundary K-matrices, which may be referred to as the fundamental representation, it is no doubt very interesting to search for non-c-number K-matrices, arising as representations in some Hilbert spaces, which may be interpreted as impurity Hilbert spaces.

We now solve (5) and (6) for $K_+(u)$ and $K_-(u)$. For the quantum R-matrix (3), One may check that the matrix $K_-(u)$ given by

$$K_-(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A_-(u) & B_-(u) \\ 0 & 0 & C_-(u) & D_-(u) \end{pmatrix} \tag{9}$$

where

$$\begin{aligned} A_-(u) &= -\frac{u^2 + 2u - 4c_a^2 - 8c_a + 4u\boldsymbol{S}_a^z}{(u - 2c_a)(u - 2c_a - 4)} \\ B_-(u) &= -\frac{4u\boldsymbol{S}_a^-}{(u - 2c_a)(u - 2c_a - 4)} \\ C_-(u) &= -\frac{4u\boldsymbol{S}_a^+}{(u - 2c_a)(u - 2c_a - 4)} \\ D_-(u) &= -\frac{u^2 + 2u - 4c_a^2 - 8c_a - 4u\boldsymbol{S}_a^z}{(u - 2c_a)(u - 2c_a - 4)} \end{aligned} \tag{10}$$

satisfies (5). Here $\boldsymbol{S}^\pm = \boldsymbol{S}^x \pm \mathrm{i}\boldsymbol{S}^y$. The matrix $K_+(u)$ can be obtained from the isomorphism of the superalgebras $\mathcal{T}_-$ and $\mathcal{T}_+$. Indeed, given a solution $\mathcal{T}_-$ of (5), then $\mathcal{T}_+(u)$ defined by

$$\mathcal{T}_+^{st}(u) = \mathcal{T}_-(-u) \tag{11}$$

is a solution of (6). The proof follows from some algebraic computations upon substituting (11) into (6) and making use of the properties of the R-matrix. Therefore, one may choose the boundary matrix $K_+(u)$ as

$$K_+(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A_+(u) & B_+(u) \\ 0 & 0 & C_+(u) & D_+(u) \end{pmatrix} \tag{12}$$

with

$$\begin{aligned} A_+(u) &= -\frac{u^2 - 2u - 4c_b^2 + 4 + 4u\boldsymbol{S}_b^z}{(u - 2c_b + 2)(u - 2c_b - 2)} \\ B_+(u) &= -\frac{4u\boldsymbol{S}_b^-}{(u - 2c_b + 2)(u - 2c_b - 2)} \\ C_+(u) &= -\frac{4u\boldsymbol{S}_b^+}{(u - 2c_b + 2)(u - 2c_b - 2)} \\ D_+(u) &= -\frac{u^2 - 2u - 4c_b^2 + 4 - 4u\boldsymbol{S}_b^z}{(u - 2c_b + 2)(u - 2c_b - 2)}. \end{aligned} \tag{13}$$

Now it can be shown that Hamiltonian (1) is related to the second derivative of the boundary transfer matrix $\tau(u)$ with respect to the spectral parameter u at $u = 0$ (up to an unimportant additive constant)

$$\begin{aligned} H = \frac{\tau''(0)}{4(V + 2W)} &= \sum_{j=1}^{L-1} H_{j,j+1} + \frac{1}{2}\overset{1}{K'}_-(0) + \frac{1}{2(V + 2W)}\Big[str_0\Big(\overset{0}{K}_+(0)G_{L0}\Big) \\ &\quad + 2\, str_0\Big(\overset{0}{K'}_+(0)H_{L0}\Big) + str_0\Big(\overset{0}{K}_+(0)(H_{L0})^2\Big)\Big] \end{aligned} \tag{14}$$

where

$$\begin{aligned} &V = str_0 K'_+(0) && W = str_0\Big(\overset{0}{K}_+(0)H^R_{L0}\Big) \\ &H_{i,j} = P_{i,j}R'_{i,j}(0) && G_{i,j} = P_{i,j}R''_{i,j}(0). \end{aligned} \tag{15}$$

This implies that the model under study admits an infinite number of conserved currents which are in involution with each other, thus assuring its integrability.

The Bethe ansatz equations may be derived using the algebraic Bethe ansatz method [19, 23, 24],

$$\begin{aligned} &\left(\frac{u_j - 1}{u_j + 1}\right)^{2L} = \prod_{\substack{i=1 \\ i\neq j}}^{N} \frac{(u_j - u_i - 2)(u_j + u_i - 2)}{(u_j - u_i + 2)(u_j + u_i + 2)} \prod_{\alpha=1}^{M_1} \frac{(u_j - v_\alpha + 1)(u_j + v_\alpha + 1)}{(u_j - v_\alpha - 1)(u_j + v_\alpha - 1)} \\ &\prod_{g=a,b} \frac{c_g + v_\alpha/2 + 1}{c_g - v_\alpha/2 + 1} \prod_{j=1}^{N} \frac{(v_\alpha - u_j + 1)(v_\alpha + u_j + 1)}{(v_\alpha - u_j - 1)(v_\alpha + u_j - 1)} = \prod_{\gamma=1}^{M_2} \frac{(v_\alpha - w_\gamma + 1)(v_\alpha + w_\gamma + 1)}{(v_\alpha - w_\gamma - 1)(v_\alpha + w_\gamma - 1)} \\ &\prod_{g=a,b} \frac{c_g - w_\gamma/2 + \frac{1}{2}}{c_g - w_\gamma/2 - \frac{1}{2}} \frac{c_g + w_\gamma/2 - \frac{1}{2}}{c_g + w_\gamma/2 + \frac{1}{2}} \prod_{\alpha=1}^{M_1} \frac{(w_\gamma - v_\alpha - 1)}{(w_\gamma - v_\alpha + 1)} \frac{(w_\gamma + v_\alpha - 1)}{(w_\gamma + v_\alpha + 1)} \\ &\qquad = \prod_{\substack{\delta=1 \\ \delta\neq\gamma}}^{M_2} \frac{(w_\gamma - w_\delta - 2)}{(w_\gamma - w_\delta + 2)} \frac{(w_\gamma + w_\delta - 2)}{(w_\gamma + w_\delta + 2)} \end{aligned} \tag{16}$$

with the corresponding energy eigenvalue E of the model

$$E = -\sum_{j=1}^{N} \frac{4}{u_j^2 - 1}. \tag{17}$$

In conclusion, we have studied an integrable Kondo problem describing two impurities coupled to the 1D supersymmetric extended Hubbard open chain. The quantum integrability of the system follows from the fact that the Hamiltonian may be embedded into a one-parameter family of commuting transfer matrices. Moreover, the Bethe ansatz equations are derived by means of the algebraic Bethe ansatz approach. It should be emphasized that the boundary K-matrices found here are highly non-trivial, since they cannot be factorized into the product of a c-number K-matrix and the local momodromy matrices. However, it is still possible to introduce a singular local monodromy matrix $\tilde{L}(u)$ and express the boundary K-matrix $K_-(u)$ as

$$K_-(u) = \tilde{L}(u)\tilde{L}^{-1}(-u) \tag{18}$$

where

$$\tilde{L}(u) = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & u + 2c_a + 2 + 2\boldsymbol{S}^z & 2\boldsymbol{S}^- \\ 0 & 0 & 2\boldsymbol{S}^+ & u + 2c_a + 2 - 2\boldsymbol{S}^z \end{pmatrix} \tag{19}$$

which constitutes a realization of the Yang–Baxter algebra (4) when ϵ tends to 0. The implication of such a singular factorization deserves further investigation. Indeed, this implies that integrable Kondo impurities discussed here appear to be, in some sense, related to a singular realization of the Yang–Baxter algebra, which in turn reflects a hidden six-vertex XXX symmetry in the original quantum R-matrix. A similar situation also occurs in the supersymmetric t–J model [19]. Also, the extension of the above construction to the case of arbitrary impurity spin is straightforward. It will be interesting to carry out the calculation of thermodynamic, equilibrium properties of the model under consideration. In particular, it is desirable to study the finite-size spectrum, which, together with the predictions of the boundary conformal field theory, will allow us to draw various critical properties. The details are deferred to a future publication.

Note added in proof. After completion of this paper, we noticed a preprint from H Frahm and N A Slavnov [25], where the method of [19] is generalized in the context of projection. We are grateful to H Frahm for bringing this reference to our attention.

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